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Evaporation from a translating drop in an electric field

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Abstract—The problem of evaporation from a spherical liquid drop translating in an electric field is investigated with perturbation solutions. The total flow field, a combination of the drop creeping translation, the electrically generated Taylor motion, and a strong non-uniform evaporation velocity assumed to be one order-of-magnitude larger than the formers, is solved by a regular perturbation technique in which the Reynolds number is used as the expansion parameter. The conservation equations of heat and mass transfer for the continuous and dispersed phases are solved by perturbation methods with the overall solution being second order accurate with respect to the perturbation variable. The results obtained from the sample calculations indicate that the electric field only alters the local temperature and concentration profiles. From the computational point of view, the current formulation is very useful, and may be used to handle a condensing drop as well as nonlinearities arising from the interactions of the thermal radiation at the drop surface. Therefore the analytical solution may easily be adopted in a large computer code which does the entire system simulation.

1. INTRODUCTION

THE EVAPORATION of liquid droplets has played a key role in many areas of technical importance. Of particular interest to this paper is the field of spray combustion where the flame structures are greatly dependent upon the rates of evaporation from the fuel drops. Due to the popular applications of fuel spray combustion, numerous theoretical and experimental studies on the vaporization process have been conducted in the past few decades. Recently, these studies have been documented in many excellent authoritative reviews, among them are those by Law [1, 2], and Sirignano [3].

In general, the evaporation from a drop is a very complex phenomenon such that any analysis would require a simultaneous description of momentum, heat, and mass transfer in the gas phase as well as in the drop phase, together with the thermodynamic couplings between the temperature and the species concentration at the drop surface. A number of models with varying level of complexities have been proposed to examine various aspects of combustion. Perhaps, the most elementary one would be the d^2-Law (see any basic textbook on combustion or one of the reviews cited above) developed for an isothermal drop held at a temperature close to the boiling point in a non-convective atmosphere. More often than not, the vaporizing drop also experiences a relative gas–liquid velocity; thus introducing a flow pattern, similar to Hill's spherical vortex, inside the drop. With the addition of an electric field of sufficient strength, the flow fields inside and outside the liquid sphere are significantly altered as a result of the charge accumulation at the gas–liquid interface. Such a flow modification is caused by the surface charges which interact with the electric field to produce electrical stresses whose tangential component is balanced by the fluid motion. The resulting velocity field is of practical interest because it can enhance the heat and mass transfer rates. Transport processes under this situation may be modeled according to the controlling resistance of the two-phase system. If the transfer resistance in the gas phase is dominant, the transport process may be considered as an external problem [4–6]. The other extreme, i.e. the internal problem, was treated by Chung *et al.* [7]. With the two phases having comparable transfer resistances, the processes have to be modeled as a conjugate problem [8–10]. All of the above studies indicate an electrical enhancement in heat and mass transfer caused by the two-phase flow/electric field interactions. For a complete discussion on a more general fundamental electrohydrodynamic effects on fluid flow, heat transfer, and combustion, the reviews by Melcher and Taylor [11], Jones [12], and Bradley [13], respectively are recommended.

To date, assessment of the electric field effects on the flame structure have only been carried out experimentally, but theoretical modeling has never been attempted. Even though promising speculations have been supported by concrete experimental evidence, the need of an understanding of the physical phenomena is crucial for proper applications. It is, therefore, the purpose of this paper to examine how the combined flow

NOMENCLATURE

C_p	heat capacitance at constant pressure
$\mathscr{D}$	mass diffusivity
E	uniform electric field strength
$\mathscr{H}_{fg}$	latent heat of vaporization
P_k	kth order Legendre polynomial
Pr	Prandtl number
$\mathscr{P}$	pressure
Q_k	kth order of Legendre polynomial of second kind
r	radial coordinate
R_g	gas constant
R	instantaneous drop radius
Sc	Schmidt number
S_σ	surface tension coefficient
T	absolute temperature
$\mathbf{U}$	velocity vector
W	molecular weight
X	mass fraction of the vaporizing species
Z	dimensionless temperature.

Greek symbols

α	thermal diffusivity
β	Stefan–Boltzmann constant
ε	Reynolds number based on streaming velocity
ε'	electric permittivity
θ	polar angle
κ	thermal conductivity
μ	dynamic viscosity
$\bar{\mu}$	cosine of polar angle
ν	kinematic viscosity
ρ	outer variable
ϱ	density
σ	electric resistivity
τ	dimensionless time
$\Phi_{\varepsilon'}$	permittivity ratio of continuous to drop phase
Φ_μ	dynamic viscosity ratio of continuous to drop phase
Φ_σ	resistivity ratio of continuous to drop phase
φ	absorptivity.

Subscripts

0	initial value
b	normal boiling point condition
E	pertaining to electric field
i	1 for continuous phase and 2 for dispersed phase
r	radial component
s	interfacial value
ss	steady state
sat	saturation state
θ	angular component
$r\theta$	shear component
∞	value far away from interface.

field, translational and electrically-generated motions, affect the transport processes during the course of evaporation.

2. PROBLEM STATEMENT

We shall consider a pure liquid drop of initial radius R_0 and at temperature T_0 travelling at the terminal velocity U_∞ in a quiescent environment at temperature T_∞ and that the two-phase system is subjected to a uniform electric field of strength E. Because $T_\infty > T_0$ heat is transported towards the drop by simultaneous convective and diffusive mechanisms. At the drop surface, a portion of the heat is consumed to sustain the vaporization process while the remainder continues to penetrate into the droplet to heat up the liquid. For the physical system so described, we now develop a transport model with the following simplifications. (i) The drop remains spherical throughout its lifetime since we are considering a drop on the order of 10 to 100 microns. The electric field tends to stretch the drop into an ellipsoidal shape with the major axis lying parallel to the electric field lines. Even so, the departure from the sphericity can only be reached to a certain degree and further deformation would make the drop unstable and it would shatter into a number of smaller droplets [14], thereby reducing the tendency of distorsion. Stewart and Morrison [15] showed that the drop deformation is of minor importance for low Reynolds number flows. (ii) The gas and liquid are Newtonian fluids and have constant physical properties recommended to be taken at the reference temperature as suggested by the simple 'one-third rule' [16]. Also, electrical conductivities of the two fluids are low enough to prevent electric and magnetic couplings. Because the flow field induced by the electrical forces is similar to that investigated by Taylor [17], limitations discussed in his paper are also carried over to this study. (iii) The flows inside and outside the drop are axisymmetric, fully developed, and are in the creeping regime. In view of the large density ratio between the liquid and the gas, the velocities are considered to be quasi-steady and that the normal component of the liquid velocity at the interface is assumed to be negligibly small [18]. (iv) The heat and mass transfer processes in the gas phase are quasi-steady, but the liquid heating is transient. This assumption is justified from the fact that the gas phase heat and mass diffusivities rates are of the order of 10^{-1}–10^{0} cm^2 s^{-1} while the liquid phase thermal diffusivity is of the order of 10^{-3} cm^2 s^{-1} and the drop surface area regression rate is approximately 10^{-4}–10^{-3} cm^2 s^{-1}. These data imply that the gas always adjusts to the changes at the boundary instantaneously.

3. METHOD OF ANALYSIS

3.1. *Fluid motion*

For a steady incompressible flow about a liquid sphere, the governing equations describing the motion of the two-phase system are given as

$$\nabla\cdot\mathbf{U} = 0 \tag{1}$$

$$\varrho\mathbf{U}\cdot\nabla\mathbf{U}+\nabla\mathscr{P} = \mu\nabla^2\mathbf{U} \tag{2}$$

which are valid for both the gas and liquid phases; however, a hat will be used to distinguish the liquid phase hereafter. The boundary conditions accompanying the above equations can be expressed as follows:

$$U_r(r\to\infty,\theta) = U_\infty\cos\theta,\quad U_\theta(r\to\infty,\theta) = -U_\infty\sin\theta \tag{3a}$$

$$\hat{U}_\theta(r\to 0,\theta)\quad\text{is finite} \tag{3b}$$

$$U_r(r = R,\theta) = u_{00}+u_{01}P_0(\cos\theta)+u_{11}P_1(\cos\theta)+u_{21}P_2(\cos\theta) \tag{3c}$$

$$\hat{U}_r(r = R,\theta) = 0 \tag{3d}$$

$$U_\theta(r = R,\theta) = \hat{U}_\theta(r = R,\theta) \tag{3e}$$

$$\mathscr{T}_{r\theta_E}(r = R,\theta)+\mathscr{T}_{r\theta}(r = R,\theta)-\hat{\mathscr{T}}_{r\theta}(r = R,\theta) = 0 \tag{3f}$$

where $P_k(\cos\theta)$ is the Legendre polynomial of degree k, $\mathscr{T}_{r\theta_E}$ is the electrical tangential stress induced by the electric field. $\mathbf{U} = (U_r, U_\theta)$ is the velocity vector, ϱ and μ are the density and dynamic viscosity respectively, and $\mathscr{P}$ is the pressure. For a full derivation of $\mathscr{T}_{r\theta_E}$, the readers are referred to the classical paper of Sir Geoffrey Taylor [17]. As shown in equation (3c), the assumed functional form of the interfacial normal velocity is composed of two parts. The leading order term u_{00} represents the potentially large evaporation velocity for a stationary drop and is assumed to be one order of magnitude larger than the droplet creeping motion. The angular variation terms account for the interfacial velocities that are induced due to droplet motion and the electric field. The P_0 and P_1 terms were shown to be adequate for a translating drop [18] while the P_2 term is added to properly model the electric field effects [19].

Similar to the method adopted in Sadhal and Ayyaswamy [18], and Chung *et al.* [20], the velocities and the pressures of the gas and the liquid phases are assumed to be in the following perturbation forms:

$$\mathbf{U} = \mathbf{U}_0+\mathbf{U}',\quad \hat{\mathbf{U}} = \hat{\mathbf{U}}',\quad \mathscr{P} = \mathscr{P}_0+\mathscr{P}',\quad \hat{\mathscr{P}} = \hat{\mathscr{P}}'. \tag{4a,b,c,d}$$

The first term on the right-hand-side of (4a), $\mathbf{U}_0$, which is independent of the drop translation, represents the uniform evaporation velocity in the radial direction as discussed above. According to (3c), $\mathbf{U}_0$ is the dominant order and the combined translation and the electric field induced velocity, $\mathbf{U}'$, is the perturbed portion. In the liquid phase only the perturbed quantity exists.

Before going into the details of the solution, let us first introduce the following dimensionless variables: $\mathbf{U}_0^* = \mathbf{U}_0/u_{00}$, $\mathbf{U}'^* = \mathbf{U}'/U_\infty$, $\hat{\mathbf{U}}'^* = \hat{\mathbf{U}}'/U_\infty$, $U_{01}P_0(\cos\theta)+U_{11}P_1(\cos\theta)+U_{21}P_2(\cos\theta) = \{u_{01}P_0(\cos\theta)+u_{11}P_1(\cos\theta)+u_{21}P_2(\cos\theta)\}/U_\infty$, $r^* = r/R$, $\varepsilon = (U_\infty R)/\nu$, $\hat{\varepsilon} = (U_\infty R)/\hat{\nu}$, $U_{00} = u_{00}R/\nu$, $\Phi_\mu = \mu/\hat{\mu}$, $\mathscr{P}_0^* = \mathscr{P}_0/(u_{00}\mu/R)$, $\mathscr{P}'^* = \mathscr{P}'/(U_\infty\mu/R)$, $\hat{\mathscr{P}}'^* = \hat{\mathscr{P}}'/(U_\infty\hat{\mu}/R)$, $\nabla^* = R\nabla$, $\mathscr{P}^* = (\mathscr{P}R^2)/(\nu\mu)$, and $\hat{\mathscr{P}}^* = (\hat{\mathscr{P}}R^2)/\hat{\nu}\hat{\mu}$. The main idea here is to scale the leading and perturbed terms of the velocity by u_{00}, and U_∞, respectively. Following Nguyen and Chung [19], the dependent quantities are now expanded as

$$\begin{aligned}\mathbf{U} &= U_{00}\mathbf{U}_0+\varepsilon\mathbf{U}'\\ &= U_{00}\mathbf{U}_0+\varepsilon\mathbf{U}_1+\varepsilon^2\mathbf{U}_2+\cdots\end{aligned} \tag{5a}$$

$$\begin{aligned}\hat{\mathbf{U}} &= \hat{\varepsilon}\hat{\mathbf{U}}'\\ &= \hat{\varepsilon}\hat{\mathbf{U}}_1+\hat{\varepsilon}^2\hat{\mathbf{U}}_2+\cdots\end{aligned} \tag{5b}$$

where $\mathbf{U}_0 = (U_{0r}, U_{0\theta})$, $\mathbf{U}_1 = (U_{1r}, U_{1\theta})$, $\hat{\mathbf{U}}_1 = (\hat{U}_{1r}, \hat{U}_{1\theta})$, etc. Note that asterisks have been omitted for convenience.

Inserting these into equations (1) and (2) and equating terms of the same powers in ε leads to a series of differential equations. While the leading order velocity in the gas phase is simply $\mathbf{U}_0 = (1/r^2, 0)$, the first order correction is obtained from the following biharmonic equations:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1-\bar{\mu}^2}{r^2}\frac{\partial^2}{\partial \bar{\mu}^2}\right]^2 \Psi_1 = \frac{U_{00}}{r^2}\left(\frac{\partial}{\partial r} - \frac{2}{r}\right)\left[\frac{\partial^2}{\partial r^2} + \frac{1-\bar{\mu}^2}{r^2}\frac{\partial^2}{\partial \bar{\mu}^2}\right]\Psi_1 \tag{6a}$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1-\bar{\mu}^2}{r^2}\frac{\partial^2}{\partial \bar{\mu}^2}\right]^2 \hat{\Psi}_1 = 0 \tag{6b}$$

where $\bar{\mu} = \cos\theta$ and Ψ is the dimensionless stream function which is related to the velocity components by

$$U_{1r} = -\frac{1}{r^2}\frac{\partial \Psi_1}{\partial \bar{\mu}}, \quad U_{1\theta} = -\frac{1}{r(1-\bar{\mu}^2)^{1/2}}\frac{\partial \Psi_1}{\partial r} \tag{7a,b}$$

subjected to

$$\Psi_1(\infty, \bar{\mu}) = \tfrac{1}{2}r^2(1-\bar{\mu}^2) \tag{8a}$$

$$-\frac{\partial \Psi_1}{\partial \bar{\mu}}(1, \bar{\mu}) = U_{01}P_0(\bar{\mu}) + U_{11}P_1(\bar{\mu}) + U_{21}P_2(\bar{\mu}) \tag{8b}$$

$$\frac{\partial \hat{\Psi}_1}{\partial \bar{\mu}}(1, \bar{\mu}) = 0 \tag{8c}$$

$$\frac{\partial \Psi_1}{\partial r}(1, \bar{\mu}) = \frac{\partial \hat{\Psi}_1}{\partial r}(1, \bar{\mu}) \tag{8d}$$

$$\Phi_\mu\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi_1}{\partial r}\right) + \frac{1-\bar{\mu}^2}{r^3}\frac{\partial^2 \Psi_1}{\partial \bar{\mu}^2}\right]\Bigg|_{r=1} - \left[-r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \hat{\Psi}_1}{\partial r}\right) + \frac{1-\bar{\mu}^2}{r^3}\frac{\partial^2 \hat{\Psi}_1}{\partial \bar{\mu}^2}\right]\Bigg|_{r=1} = \tfrac{5}{2}\mathscr{W}Q_2(\mu) \tag{8e}$$

$$\frac{1}{r^2}\hat{\Psi}_1(0, \bar{\mu}) \quad \text{is finite} \tag{8f}$$

where the quantity Φ_μ stands for the dynamic viscosity ratio of the continuous to the dispersed phase, and is likewise for other physical properties. The parameter $\mathscr{W}$ represents a measure of the relative importance of the electric field to the translating motion of the drop, i.e.

$$\mathscr{W} = \frac{4V_{cr}(1+\Phi_\mu)}{U_\infty} \tag{9a}$$

and

$$V_{cr} = \frac{9\hat{\varepsilon}' RE^2(1-\Phi_{\varepsilon'}\Phi_\sigma)}{10\hat{\mu}(\Phi_\mu+1)(2+\Phi_\sigma)^2} \tag{9b}$$

where V_{cr} is the maximum speed generated by the electric field in the absence of translation, E is the electric field strength, ε' and σ are the electrical permittivity and resistivity, respectively. An inspection of the expression for $\mathscr{W}$ indicates that it can be either positive or negative depending on whether the direction of internal circulation is from the pole to the equator $(+V_{cr})$ or from the equator to the pole $(-V_{cr})$. These two possible directions are dictated by the electrical properties of the two phases such that $\Phi_\varepsilon\Phi_\sigma < 1$ for the former, and greater than unity for the latter case.

As in the Galerkin method, the solution to equations (6a) and (6b) subjected to boundary conditions (8a)–(8f) is now approximated as:

$$\Psi_1 = -U_{01}Q_0(\bar{\mu}) + \left(\frac{1}{2}r^2 + \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3} + \frac{A_4}{r^4}\right)Q_1(\bar{\mu}) + \left(\frac{A_5}{r^2} + \frac{A_6}{r^3} + \frac{A_7}{r^4} + \frac{A_8}{r^5}\right)Q_2(\bar{\mu}) \tag{10}$$

$$\hat{\Psi}_1 = (\hat{A}_1 r^2 + \hat{A}_2 r^4)Q_1(\bar{\mu}) + (\hat{A}_3 r^3 + \hat{A}_4 r^5)Q_2(\bar{\mu}) \tag{11}$$

where the function $Q_k(\mu)$ is defined as

$$Q_k(\bar{\mu}) = \begin{cases} \bar{\mu}, & k = 0 \\ -2\int_{-1}^{\bar{\mu}} P_k(\mu^*)\,\mathrm{d}\mu^*, & k \neq 0 \end{cases} \tag{12}$$

in which the coefficients in equations (10) and (11) are found from the boundary conditions and the additional four Galerkin conditions. The first two require that the solution be error free on the surface of the drop while the second two require that errors be distributed evenly throughout the continuous phase. After considerable manipulation, one obtains the following system of algebraic equations:

$$2A_1+2A_2+2A_3+2A_4 = U_{11}-1 \tag{13}$$

$$2A_5+2A_6+2A_7+2A_8 = U_{21} \tag{14}$$

$$\hat{A}_1+\hat{A}_2 = 0 \tag{15}$$

$$\hat{A}_3+\hat{A}_4 = 0 \tag{16}$$

$$A_1+2A_2+3A_3+4A_4+2\hat{A}_2 = 1 \tag{17}$$

$$2A_5+3A_6+4A_7+5A_8+2\hat{A}_4 = 0 \tag{18}$$

$$\Phi_\mu(3A_1+6A_2+10A_3+15A_4)-3\hat{A}_2 = 0 \tag{19}$$

$$\Phi_\mu(3A_5+7A_6+12A_7+18A_8)-5\hat{A}_4 = -\tfrac{5}{4}\mathscr{W}-\tfrac{5}{2}\Phi_\mu U_{21} \tag{20}$$

$$A_2(72+24U_{00})+A_3(280+70U_{00})+A_4(720+144U_{00}) = 0 \tag{21}$$

$$A_6(144+42U_{00})+A_7(504+112U_{00})+A_8(1200+216U_{00}) = 0 \tag{22}$$

$$A_2(12+\tfrac{24}{7}U_{00})+A_3(40+\tfrac{35}{4}U_{00})+A_4(90+16U_{00}) = 0 \tag{23}$$

$$A_6(18+\tfrac{42}{9}U_{00})+A_7(56+\tfrac{56}{5}U_{00})+A_8(120+\tfrac{216}{11}U_{00}) = 0. \tag{24}$$

Once the constants are determined, the velocities can be computed using (7a) and (7b). The results are

$$U_r(r,\theta) = \frac{U_{00}}{r^2}P_0(\bar{\mu})+\varepsilon\left\{\frac{U_{01}}{r^2}P_0(\bar{\mu})+2\left(\frac{1}{2}+\frac{A_1}{r^3}+\frac{A_2}{r^4}+\frac{A_3}{r^5}+\frac{A_4}{r^6}\right)P_1(\bar{\mu})+2\left(\frac{A_5}{r^4}+\frac{A_6}{r^5}+\frac{A_7}{r^6}+\frac{A_8}{r^7}\right)P_2(\bar{\mu})\right\} \tag{25}$$

$$U_\theta(r,\theta) = -\varepsilon\left\{\left(1-\frac{A_1}{r^3}-2\frac{A_2}{r^4}-3\frac{A_3}{r^5}-4\frac{A_4}{r^6}\right)P_1^1(\bar{\mu})-\left(2\frac{A_5}{r^4}+3\frac{A_6}{r^5}+4\frac{A_7}{r^6}+5\frac{A_8}{r^7}\right)P_2^1(\bar{\mu})\right\} \tag{26}$$

$$\hat{U}_r(r,\theta) = \hat{\varepsilon}\{\hat{A}_1(2-2r^2)P_1(\bar{\mu})+2\hat{A}_3(r-r^3)P_2(\bar{\mu})\} \tag{27}$$

$$\hat{U}_\theta(r,\theta) = -\hat{\varepsilon}\{\hat{A}_1(2-4r^2)P_1^1(\bar{\mu})-\hat{A}_3(3r+5r^3)P_2^1(\bar{\mu})\} \tag{28}$$

where $P_k^1(\bar{\mu})$s have been defined slightly different from the associated Legendre functions of degree k and order 1 as

$$P_k^1(\bar{\mu}) = \left[\frac{(1-\bar{\mu}^2)^{1/2}}{2k-1}\right]\frac{\mathrm{d}P_k(\bar{\mu})}{\mathrm{d}\bar{\mu}}. \tag{29}$$

Knowing the velocity, the drag force acting on the liquid drop can be obtained. For a droplet experiencing an interfacial mass flux, the total drag is the sum of three contributions: the skin friction forces, the pressure forces, and the forces induced by the momentum flux. Because the derivation is straightforward, there is no need to repeat it here, only the final result is given,

$$F_D = \tfrac{2}{9}[4U_{11}-4-8A_1-16A_2-38A_3-80A_4-U_{00}(U_{11}-4A_1-10A_2-18A_3-28A_4+1)] \tag{30}$$

which has been normalized with the value of the solid sphere, $3\pi\mu U_\infty D$.

3.2. *Heat and mass transfer in the continuous phase*

The energy and the species equations in the gas phase are written in terms of the dimensionless temperature, $Z = (T-T_\infty)/(T_0-T_\infty)$, and the mass fraction X as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial Z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\bar{\mu}}\left[(1-\bar{\mu}^2)\frac{\partial Z}{\partial\bar{\mu}}\right] = \varepsilon\,Pr\left[U_r\frac{\partial Z}{\partial r}-\frac{U_\theta}{r}(1-\bar{\mu}^2)^{1/2}\frac{\partial Z}{\partial\bar{\mu}}\right] \tag{31}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial X}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\bar{\mu}}\left[(1-\bar{\mu}^2)\frac{\partial X}{\partial\bar{\mu}}\right] = \varepsilon\,Sc\left[U_r\frac{\partial X}{\partial r}-\frac{U_\theta}{r}(1-\bar{\mu}^2)^{1/2}\frac{\partial X}{\partial\bar{\mu}}\right] \tag{32}$$

where the Pr ($=\nu/\alpha$) and Sc ($=\nu/\mathscr{D}$) are the Prandtl and Schmidt numbers, respectively. The transfer coefficients ν, α, and $\mathscr{D}$ are the momentum, thermal, and mass diffusivities, respectively. The boundary conditions of equations (31) and (32) are the followings:

$$Z(r \to \infty, \bar{\mu}) = 0 \tag{33a}$$

$$\frac{\partial Z}{\partial \bar{\mu}}(r, \bar{\mu} = \pm 1) = 0 \tag{33b}$$

$$Z(r = 1, \bar{\mu}) = Z_{00} + \varepsilon[Z_{01}P_0(\bar{\mu}) + Z_{11}P_1(\bar{\mu}) + Z_{21}P_2(\bar{\mu})] \tag{33c}$$

$$X(r \to \infty, \bar{\mu}) = 0 \tag{33d}$$

$$\frac{\partial X}{\partial \bar{\mu}}(r, \bar{\mu} = \pm 1) = 0 \tag{33e}$$

$$X(r = 1, \bar{\mu}) = X_{00} + \varepsilon[X_{01}P_0(\bar{\mu}) + X_{11}P_1(\bar{\mu}) + X_{21}P_2(\bar{\mu})]. \tag{33f}$$

Note that the interfacial temperature and concentration have been modeled in the same way as was done for the interfacial velocity. The parameters Z_{ij}s and X_{ij}s introduced in (33c) and (33f) are interelated rather than being independent and their coupling is dictated by a phase equilibrium relation such as that recommended by Hallett [21],

$$\mathscr{P}_{\text{sat}} = \exp\left[\chi_1 - \frac{\chi_2}{T_s + \chi_3}\right] \tag{34}$$

where T_s is the absolute surface temperature, $\mathscr{P}_{\text{sat}}$ is the saturation vapor pressure at the drop surface in the unit of millimeters of mercury, and χ_is are the fitting constants. As a word of caution, very small droplets, owing to the surface tension, exhibit a higher vapor pressure than bulk fluids by a factor of $(1 + WS_\sigma/R\hat{\rho}R_gT)$ where S_σ is the surface tension, and R_g is the universal gas constant and must be accounted for when necessary.

For a pure liquid drop with a molecular weight $\hat{W}$, the mass fraction at the interface is

$$\frac{1}{X_s} = 1 + \frac{W}{\hat{W}}\left(\frac{\mathscr{P}_\infty}{\mathscr{P}_{\text{sat}}} - 1\right). \tag{35}$$

Upon substitution of the interfacial temperature for T_s in equation (34), the resulting equation is then expanded for small ε to give, after combining with (35), the following relationships:

$$\frac{1}{X_{00}} = 1 + \Phi_w\left\{\mathscr{P}_\infty \exp\left[-\chi_1 + \frac{\chi_2}{Z_{00}(T_0 - T_\infty) + T_\infty + \chi_3}\right] - 1\right\} \tag{36}$$

$$X_{k1} = \frac{\chi_2(T_0 - T_\infty)[X_{00} + X_{00}^2(\Phi_w - 1)]}{[Z_{00}(T_0 - T_\infty) + T_\infty + \chi_3]^2} Z_{k1}, \quad k = 0, 1, 2. \tag{37}$$

To begin the derivation of the solutions of equations (35) and (36), we note that they are exactly identical for the case of a total analogy between heat and mass transfer (unity Lewis number). In many systems, the magnitudes of *Pr* and *Sc* are nearly the same and the Lewis number is in fact close to unity. Assuming this is true, only the energy or the species continuity equation needs to be solved; hence, the subsequent discussion is directed to the energy equation.

Following the singular perturbation procedure described in ref. [20], the inner and outer solutions, denoted by superscripts i and o respectively, are expanded as

$$Z^{\text{i}}(r, \bar{\mu}) = \sum_{k=0}^{\infty} \varepsilon^k Z_k^{\text{i}}(r, \bar{\mu}) \tag{38}$$

$$Z^{\text{o}}(\rho, \bar{\mu}, \varepsilon) = \sum_{k=0}^{\infty} F_k(\varepsilon) Z_k^{\text{o}}(\rho, \bar{\mu}) \tag{39}$$

where a strained coordinate ρ $(= \varepsilon r)$ has been introduced in the outer expansion. Again, substituting expressions (38) and (39) into equation (31), followed by extracting terms of the same power in ε yields a series of simpler problems for the determination of Z_k^{i}s and Z_k^{o}s. For the lowest order approximation, Fendell *et al.* [22] derived the following solutions

$$Z_0^{\text{i}}(r) = Z_{00}\frac{e^{-\tilde{U}_{00}/r} - 1}{e^{-\tilde{U}_{00}} - 1} \tag{40}$$

$$Z_0^{\text{o}}(\rho, \bar{\mu}) = -Z_{00}\frac{\tilde{U}_{00}\, e^{-Pr/2\rho(1-\bar{\mu})}}{\rho(e^{-\tilde{U}_{00}} - 1)} \tag{41}$$

where we have adopted the notation that $\tilde{U}_{00} = Pr\, U_{00}$. A similar definition is also assumed for other variables.

At the next door approximation, the equation becomes considerably complicated; however, a first order approximation can still be obtained with a reasonable amount of effort. The first order equation is given by

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial Z^{\mathrm{i}}_1}{\partial r}\right)-\tilde{U}_{00}\frac{\partial Z^{\mathrm{i}}_1}{\partial r}\right]+\frac{1}{r^2}\frac{\partial}{\partial\bar{\mu}}\left[(1-\bar{\mu}^2)\frac{\partial Z^{\mathrm{i}}_1}{\partial\bar{\mu}}\right]=Pr\,U_{1r}\frac{\partial Z^{\mathrm{i}}_0}{\partial r}. \tag{42}$$

In view of the functional form of the velocity, it is plausible to assume the solution to be a linear combination of the first three Legendre functions, i.e.

$$Z^{\mathrm{i}}_1(r,\bar{\mu})=\sum_{k=0}^{2}f_kP_k(\bar{\mu}). \tag{43}$$

This form of the solution transforms equation (42) into three linearly independent ordinary differential equations for which solution techniques have been well-developed,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}f_k}{\mathrm{d}r}\right)-\tilde{U}_{00}\frac{\mathrm{d}f_k}{\mathrm{d}r}-k(k+1)f_k=\begin{cases}\dfrac{U_{01}}{r^2}\left[\dfrac{\tilde{Z}_{00}\tilde{U}_{00}\,\mathrm{e}^{-\tilde{U}_{00}/r}}{\mathrm{e}^{-\tilde{U}_{00}}-1}\right], & k=0 \qquad (44)\\[2ex] \left(1+\dfrac{2}{r^2}\displaystyle\sum_{i=1}^{4}\dfrac{A_i}{r^i}\right)\left[\dfrac{\tilde{Z}_{00}\tilde{U}_{00}\,\mathrm{e}^{-\tilde{U}_{00}/r}}{\mathrm{e}^{-\tilde{U}_{00}}-1}\right], & k=1 \qquad (45)\\[2ex] \left(2r\displaystyle\sum_{i=5}^{8}\dfrac{A_i}{r^i}\right)\left[\dfrac{\tilde{Z}_{00}\tilde{U}_{00}\,\mathrm{e}^{-\tilde{U}_{00}/r}}{\mathrm{e}^{-\tilde{U}_{00}}-1}\right], & k=2 \qquad (46)\end{cases}$$

subjected to the non-homogeneous boundary condition,

$$f_k(r=1)=Z_{k1},\quad k=0,1,2. \tag{47}$$

General solutions of these equations are obtainable by using the series expansion technique to yield one solution to the corresponding homogeneous equation, whereas the other independent solution sought via the method-of-reduction of order. Once complete, the particular solution is obtained by the method of variation-of-parameters. Further details leading to the results are given elsewhere [23].

$$f_0(r)=Z_{01}+\frac{U_{01}\tilde{Z}_{00}}{1-\mathrm{e}^{-\tilde{U}_{00}}}\left(\frac{\mathrm{e}^{-\tilde{U}_{00}/r}}{r}-\mathrm{e}^{-\tilde{U}_{00}}\right)+C_0(\mathrm{e}^{-\tilde{U}_{00}/r}-\mathrm{e}^{-U_{00}}) \tag{48}$$

$$\begin{aligned}f_1(r)=\mathrm{e}^{-\tilde{U}_{00}/r}\left(r+\frac{\tilde{U}_{00}}{2}\right)[H(r)-H(1)]+\left(r-\frac{\tilde{U}_{00}}{2}\right)\left[G(r)-G(1)+\frac{2Z_{11}}{2-\tilde{U}_{00}}\right]\\ +C_1\left[\left(r+\frac{\tilde{U}_{00}}{2}\right)\mathrm{e}^{-\tilde{U}_{00}/r}-\left(r-\frac{\tilde{U}_{00}}{2}\right)\frac{2+\tilde{U}_{00}}{2-\tilde{U}_{00}}\mathrm{e}^{-\tilde{U}_{00}}\right]\end{aligned} \tag{49}$$

$$\begin{aligned}f_2(r)=(C_2-Q(1))(\tilde{U}^2_{00}-6r\tilde{U}_{00}+12r^2)\int_1^r\frac{\mathrm{e}^{-\tilde{U}_{00}/\xi}\,\mathrm{d}\xi}{\xi^2(\tilde{U}^2_{00}-6\xi\tilde{U}_{00}+12\xi^2)^2}\\ +(\tilde{U}^2_{00}-6r\tilde{U}_{00}+12r^2)\left(V(r)+\frac{Z_{21}}{\tilde{U}^2_{00}-6\tilde{U}_{00}+12}\right)\end{aligned} \tag{50}$$

where G, H, Q, and V are functions of the radial coordinate, and are defined by

$$\begin{aligned}G(r)=\frac{4\tilde{Z}_{00}\,\mathrm{e}^{-\tilde{U}_{00}/r}}{\tilde{U}^2_{00}(\mathrm{e}^{-\tilde{U}_{00}}-1)}\left[\frac{r^2}{2}+A_1\left(\frac{1}{r}+\frac{3}{\tilde{U}_{00}}\right)+A_2\left(\frac{1}{r^2}+\frac{4}{r\tilde{U}_{00}}+\frac{4}{\tilde{U}^2_{00}}\right)\right.\\ \left.+A_3\left(\frac{1}{r^3}+\frac{5}{r^2\tilde{U}_{00}}+\frac{10}{r\tilde{U}^2_{00}}+\frac{10}{\tilde{U}^3_{00}}\right)+A_4\left(\frac{1}{r^4}+\frac{6}{r^3\tilde{U}_{00}}+\frac{18}{r^2\tilde{U}^2_{00}}+\frac{36}{r\tilde{U}^3_{00}}+\frac{36}{\tilde{U}^4_{00}}\right)\right]\end{aligned} \tag{51}$$

$$H(r)=\frac{4\tilde{Z}_{00}}{\tilde{U}^2_{00}(\mathrm{e}^{-\tilde{U}_{00}}-1)}\left[-\frac{r}{2}(r-\tilde{U}_{00})+\frac{A_1}{r^2}\left(2r-\frac{\tilde{U}_{00}}{2}\right)+\frac{A_2}{r^3}\left(r-\frac{\tilde{U}_{00}}{3}\right)+\frac{A_3}{r^4}\left(\frac{2}{3}r-\frac{\tilde{U}_{00}}{4}\right)+\frac{A_4}{r^5}\left(\frac{r}{2}-\frac{\tilde{U}_{00}}{5}\right)\right] \tag{52}$$

$$Q(r) = \frac{2\tilde{Z}_{00}\tilde{U}_{00}}{1-\mathrm{e}^{-\tilde{U}_{00}}}\left[A_5\left(\frac{\tilde{U}_{00}^2}{3r^3}-3\frac{\tilde{U}_{00}}{r^2}+\frac{12}{r}\right)+A_6\left(\frac{\tilde{U}_{00}^2}{4r^4}-2\frac{\tilde{U}_{00}}{r^3}+\frac{6}{r^2}\right)\right.$$
$$\left.+A_7\left(\frac{\tilde{U}_{00}^2}{5r^5}-\frac{3\tilde{U}_{00}}{2r^4}+\frac{4}{r^3}\right)+A_8\left(\frac{\tilde{U}_{00}^2}{6r^6}-\frac{6\tilde{U}_{00}}{5r^5}+\frac{3}{r^4}\right)\right] \tag{53}$$

$$V(r) = \int_1^r \frac{Q(\xi)\,\mathrm{e}^{-\tilde{U}_{00}/\xi}\,\mathrm{d}\xi}{\xi^2(\tilde{U}_{00}^2-6\xi\tilde{U}_{00}+12\xi^2)^2} \tag{54}$$

where C_0, C_1, and C_2 are the constants and remain to be determined from the matching asymptotic condition,

$$Z_0^{\mathrm{i}}(r\to\infty,\bar{\mu})+\varepsilon Z_1^{\circ}(r\to\infty,\bar{\mu})+O(\varepsilon^2) = F_0(\varepsilon)Z^{\circ}(\rho\to 0,\bar{\mu})+O(F_1(\varepsilon)). \tag{55}$$

By using the procedure outlined by Van Dyke [24], the coefficients are found to be

$$C_0 = 2\frac{U_{01}\tilde{Z}_{00}\,\mathrm{e}^{-\tilde{U}_{00}}-\tilde{U}_{00}\tilde{Z}_{00}}{2(\mathrm{e}^{-\tilde{U}_{00}}-1)^2}+\frac{Z_{01}}{\mathrm{e}^{-\tilde{U}_{00}}-1} \tag{56}$$

$$C_1 = \frac{\tilde{U}_{00}-2}{2-\tilde{U}_{00}-(2+\tilde{U}_{00})\,\mathrm{e}^{-\tilde{U}_{00}}}\left[\frac{\tilde{Z}_{00}}{\mathrm{e}^{-\tilde{U}_{00}}-1}\left(1+12\frac{A_1}{\tilde{U}_{00}^3}+16\frac{A_2}{\tilde{U}_{00}^4}+40\frac{A_3}{\tilde{U}_{00}^5}+144\frac{A_4}{\tilde{U}_{00}^6}\right)\right.$$
$$\left.+\frac{2Z_{11}}{2-\tilde{U}_{00}}-H(1)-G(1)\right] \tag{57}$$

$$C_2 = Q(1)+\frac{\displaystyle\int_0^1\frac{\xi^4 Q\left(\frac{1}{\xi}\right)\mathrm{e}^{-\tilde{U}_{00}\xi}\,\mathrm{d}\xi}{(\tilde{U}_{00}^2\xi^2-6\tilde{U}_{00}\xi+12)^2}+\frac{Z_{21}}{(\tilde{U}_{00}^2-6\tilde{U}_{00}+12)}}{\displaystyle\int_0^1\frac{-\xi^4\,\mathrm{e}^{-\tilde{U}_{00}\xi}\,\mathrm{d}\xi}{(\tilde{U}_{00}^2\xi^2-6\tilde{U}_{00}\xi+12)^2}}. \tag{58}$$

It should be noted that $F_0(\varepsilon)$ has been chosen to be ε so that the above matching process is accurate up to the first order in ε. The constant C_2 requires the evaluation of definite integrals which can be done by means of Gaussian quadrature of the moments.

3.3. *Heat transfer in the drop interior*

In the dispersed phase, the liquid heating process is described by the unsteady convective–diffusion equation,

$$\frac{\partial\hat{Z}}{\partial\tau}+\varepsilon\Phi_v\,\hat{P}r\left[\hat{U}_r\frac{\partial\hat{Z}}{\partial r}-\frac{\hat{U}_\theta}{r}(1-\bar{\mu}^2)^{1/2}\frac{\partial\hat{Z}}{\partial\bar{\mu}}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\hat{Z}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\bar{\mu}}\left[(1-\bar{\mu}^2)\frac{\partial\hat{Z}}{\partial\bar{\mu}}\right] \tag{59}$$

where τ $(=\int_0^t \hat{\alpha}/R^2\,\mathrm{d}t')$ is the dimensionless time. The initial and boundary conditions are

$$\hat{Z}(\tau = 0, r, \bar{\mu}) = 1 \tag{60a}$$

$$\hat{Z}(\tau, r = 1, \bar{\mu}) = Z_{00}+\varepsilon[Z_{01}P_0(\bar{\mu})+Z_{11}P_1(\bar{\mu})+Z_{21}P_2(\bar{\mu})] \tag{60b}$$

$$\frac{\partial\hat{Z}}{\partial r}(\tau, r\to 0, \bar{\mu}) = 0 \tag{60c}$$

$$\frac{\partial\hat{Z}}{\partial\bar{\mu}}(\tau, r, \bar{\mu} = \pm 1) = 0. \tag{60d}$$

The above equations do not possess any singularities; hence, the regular perturbation technique is well-suited. Owing to this feature, the interior temperature is now sought in the form

$$\hat{Z}(\tau, r, \bar{\mu}) = \hat{Z}_0(\tau, r)+\varepsilon\hat{Z}_1(\tau, r, \bar{\mu})+O(\varepsilon^2). \tag{61}$$

Inserting this truncated series into the governing equation yields sets of equations to be solved for $\hat{Z}_0$ and $\hat{Z}_1$. For the first two leading approximations, the equations for $\hat{Z}_0$ and $\hat{Z}_1$ are

$$\frac{\partial\hat{Z}_0}{\partial\tau} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\hat{Z}_0}{\partial r}\right) \tag{62}$$

$$\frac{\partial \hat{Z}_1}{\partial \tau}+\Phi_v\, Pr\, \hat{U}_r \frac{\partial \hat{Z}_0}{\partial r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \hat{Z}_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \bar{\mu}}\left[(1-\bar{\mu}^2)\frac{\partial \hat{Z}_1}{\partial \bar{\mu}}\right]. \tag{63}$$

The initial and boundary conditions become

$$\hat{Z}_0(\tau=0,r)=1 \tag{64a}$$

$$\hat{Z}_0(\tau,r=1)=Z_{00} \tag{64b}$$

$$\frac{\partial \hat{Z}_0}{\partial r}(\tau,r\to 0)=0 \tag{64c}$$

$$\hat{Z}_1(\tau=0,r,\bar{\mu})=0 \tag{65a}$$

$$\hat{Z}_1(\tau,r=1,\bar{\mu})=Z_{01}P_0(\bar{\mu})+Z_{11}P_1(\bar{\mu})+Z_{21}P_2(\bar{\mu}) \tag{65b}$$

$$\frac{\partial \hat{Z}_1}{\partial r}(\tau,r\to 0,\bar{\mu})=0. \tag{65c}$$

It is appropriate to add a remark here that the boundary conditions are time-dependent and the solution procedure consists of two steps: first, the method of separation of variables is utilized to get the solution to the auxiliary problem. Next, the solution to the present problem with time-varying boundary condition is obtained from the corresponding auxiliary solution by an application of Duhamel's theorem. Because the solution process is lengthy and tedious, the readers are referred to ref. [23] for the missing details. Thus, the results are summarized as follows:

$$\hat{Z}_0(\tau,r)=Z_{00}+2\sum_{k=1}^{\infty}(-1)^k\frac{\sin\beta_k r}{\beta_k r}\left[\frac{\partial}{\partial\tau}\int_0^{\tau}(Z_{00}-1)\,\mathrm{e}^{-\beta_k^2(\tau-\lambda)}\,\mathrm{d}\lambda\right] \tag{66}$$

$$\hat{Z}_1(\tau,r,\bar{\mu})=\sum_{k=0}^{NT}\hat{f}_k(r)P_k(\bar{\mu}) \tag{67}$$

where

$$\hat{f}_0(\tau,r)=Z_{01}(\tau)+2\sum_{k=1}^{\infty}(-1)^k\frac{\sin\beta_k r}{\beta_k r}\left[\frac{\partial}{\partial\tau}\int_0^{\tau}Z_{01}\,\mathrm{e}^{-\beta_k^2(\tau-\lambda)}\,\mathrm{d}\lambda\right] \tag{68}$$

$$\hat{f}_1(\tau,r)=Z_{11}r+\frac{1}{r}\hat{G}_{ss}(\tau,r)+4\Phi_v\,\hat{P}r\,\hat{S}(\tau,r)-\sum_{k=1}^{\infty}\frac{\hat{C}_k}{\sqrt{r}}J_{3/2}(\omega_k r)\frac{\partial}{\partial\tau}\int_0^{\tau}Z_{11}(\lambda)\,\mathrm{e}^{-\omega_k^2(\tau-\lambda)}\,\mathrm{d}\lambda \tag{69}$$

with

$$\begin{aligned}\hat{G}_{ss}(\lambda,r)=-4\Phi_v\,\hat{P}r\sum_{k=1}^{\infty}(-1)^k&\left[\left(r^2-\left(1+\frac{10}{\beta_k^2}\right)\right)\frac{\cos\beta_k r}{\beta_k^2}+\left(\left(1+\frac{10}{\beta_k^2}\right)\frac{1}{r}-5r\right)\frac{\sin\beta_k r}{\beta_k^3}+(-1)^k\frac{10}{\beta_k^4}r^2\right]\\&\times\left[\hat{A}_1\left(\mathrm{e}^{-\beta_k^2\lambda}-\frac{\partial}{\partial\lambda}\int_0^{\lambda}Z_{00}(\lambda^*)\,\mathrm{e}^{-\beta_k^2(\lambda-\lambda^*)}\,\mathrm{d}\lambda^*\right)\right]\end{aligned} \tag{70}$$

$$\begin{aligned}\hat{S}(\tau,r)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n J_{3/2}(\omega_k r)}{N_{1k}\sqrt{r}}\hat{V}(\tau,\hat{A}_1,\beta_n,\omega_k)\int_0^1&\left[\left(r^2-\left(1+\frac{10}{\beta_n^2}\right)\right)\frac{\cos\beta_n r}{\beta_n^2}\right.\\&\left.+\left(\left(1+\frac{10}{\beta_n^2}\right)\frac{1}{r}-5r\right)\frac{\sin\beta_n r}{\beta_n^3}+(-1)^n\frac{10}{\beta_n^4}r^2\right]\sqrt{r}J_{3/2}(\omega_k r)\,\mathrm{d}r\end{aligned} \tag{71}$$

$$\hat{C}_k=\frac{1}{N_{1k}}\int_0^1 r^{5/2}J_{3/2}(\omega_k r)\,\mathrm{d}r \tag{72}$$

$$N_{1k}=\int_0^1 rJ_{3/2}^2(\omega_k r)\,\mathrm{d}r \tag{73}$$

and

$$\hat{f}_2(\tau,r)=Z_{21}r^2+\frac{1}{r}\hat{Q}_{ss}(\tau,r)+4\Phi_v\,\hat{P}r\,\hat{D}(\tau,r)-\sum_{k=1}^{\infty}\frac{\hat{E}_k}{\sqrt{r}}J_{5/2}(\gamma_k r)\frac{\partial}{\partial\tau}\int_0^{\tau}Z_{21}(\lambda)\,\mathrm{e}^{-\gamma_k^2(\tau-\lambda)}\,\mathrm{d}\lambda \tag{74}$$

with

$$\hat{Q}_{ss}(\lambda, r) = -4\Phi_v\, \hat{P}r\, \hat{A}_3 \sum_{k=1}^{\infty} (-1)^k \left[\left(r^3 - \left(1 + \frac{28}{\beta_k^2}\right) r + \left(\frac{6}{\beta_k^2} + \frac{168}{\beta_k^4}\right)\frac{1}{r}\right)\frac{\cos \beta_k r}{\beta_k^2} - \left(\left(\frac{6}{\beta_k^2} + \frac{168}{\beta_k^4}\right)\frac{1}{r^2} \right.\right.$$

$$\left.\left. - \left(3 + \frac{84}{\beta_k^2}\right) + 7r^2 \right)\frac{\sin \beta_k r}{\beta_k^3} + (-1)^k \frac{(12\beta_k^2 - 168)r^3}{\beta_k^6} \right] \left(e^{-\beta_k^2 \lambda} - \frac{\partial}{\partial \lambda}\int_0^{\lambda} Z_{00}(\lambda^*)\, e^{-\beta_k^2(\lambda - \lambda^*)}\, d\lambda^* \right) \quad (75)$$

$$\hat{D}(\tau, r) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^n J_{5/2}(\gamma_k r)}{N_{2k}\sqrt{r}}\, \hat{V}(\tau, \hat{A}_3, \beta_n, \gamma_k) \int_0^1 \left[\left(r^3 - \left(1 + \frac{18}{\beta_n^2}\right) r + \left(\frac{6}{\beta_n^2} + \frac{168}{\beta_n^4}\right)\frac{1}{r}\right)\frac{\cos \beta_n r}{\beta_n^2} \right.$$

$$\left. - \left(\left(\frac{6}{\beta_n^2} + \frac{168}{\beta_n^4}\right)\frac{1}{r^2} - \left(3 + \frac{84}{\beta_n^2}\right) + 7r^2 \right)\frac{\sin \beta_n r}{\beta_n^3} + (-1)^n \frac{(12\beta_n^2 - 168)r^3}{\beta_n^6} \right] \sqrt{r} J_{5/2}(\gamma_k r)\, dr \quad (76)$$

$$\hat{E}_k = \frac{1}{N_{2k}}\int_0^1 r^{7/2} J_{5/2}(\gamma_k r)\, dr \quad (77)$$

$$N_{2k} = \int_0^1 r J_{5/2}^2(\gamma_k r)\, dr \quad (78)$$

where $\beta_k = k\pi$. ω_k and γ_k are the positive roots of $J_{3/2}(\omega_k) = 0$ and $J_{5/2}(\gamma_k) = 0$ respectively.

A quick inspection indicates that the above analysis is accurate to ε^2. With that in mind, the time-dependent temperature distribution can be calculated at any location throughout the two-phase system once the surface temperature is specified. Since it is not known a priori, it deserves to be addressed in the next section.

3.4. *Droplet time history*

One of the most useful parameters in droplet evaporation is its regression history. To obtain that information, the interfacial velocity is now related to the mass fraction. In terms of weighted mass fraction, it is expressed as

$$U_s = -\frac{\mathscr{D}}{\nu}\frac{\partial X}{\partial r}(r = 1, \theta) \quad (79)$$

where the subscript s pertains to the value at the drop surface. With the aid of the mass flux obtained from the inner solution, the resulting expression is then arranged in powers of ε to give

$$U_{00} = -\frac{\mathscr{D}}{\nu}\left[\frac{X_{00}\tilde{U}_{00}\, e^{-\tilde{U}_{00}}}{e^{-\tilde{U}_{00}} - 1}\right] \quad (80)$$

$$U_{01} = -\frac{\mathscr{D}}{\nu}\left[C_{0x}\tilde{U}_{00} + \frac{U_{01}\tilde{X}_{00}}{1 - e^{-\tilde{U}_{00}}}(\tilde{U}_{00} - 1) \right] e^{-\tilde{U}_{00}} \quad (81)$$

$$U_{11} = -\frac{\mathscr{D}}{\nu}\left[\frac{2X_{11}}{2 - \tilde{U}_{00}} - \frac{C_{1x}\tilde{U}_{00}^3\, e^{-\tilde{U}_{00}}}{2(2 - \tilde{U}_{00})} \right] \quad (82)$$

$$U_{21} = -\frac{\mathscr{D}}{\nu}\left[\frac{(24 - 6\tilde{U}_{00})X_{21} + C_{2x}\, e^{-\tilde{U}_{00}}}{\tilde{U}_{00}^2 - 6\tilde{U}_{00} + 12} \right] \quad (83)$$

where an extra subscript x added to C_i has been used to denote coefficients of the concentration profile. Another independent surface condition is secured by requiring the continuity of the local energy flux across the dispersed-continuous interface stated in mathematical form as

$$-\varrho_s \mathscr{D}\mathscr{H}_{fg}\frac{\partial X}{\partial r}(\tau, r = 1) = (T_0 - T_\infty)\left[\kappa\frac{\partial Z}{\partial r}(\tau, r = 1) - \hat{\kappa}\frac{\partial \hat{Z}}{\partial r}(\tau, r = 1) \right] + \frac{\beta}{R_\varphi}[T_\infty^4 - (Z_s(T_0 - T_\infty) + T_\infty)^4] \quad (84)$$

where φ, κ, ϱ_s, and β are the absorptivity, the thermal conductivity, the density, and the Stefan–Boltzmann constant, respectively. R_φ is actually equivalent to the total thermal resistance, and is equal to $(1 - \hat{\varphi})/\hat{\varphi} + 1/\varphi$. Because the latent heat of vaporization is generally a weak function of temperature, $\mathscr{H}_{fg}$ is taken to be constant. With regard to the thermal radiation, we have treated the drop as an opaque spherical particle exchanging radiant energy with the ambient gray gas so that the portion of the irradiation which is not absorbed is, therefore, reflected back into the surrounding. Obviously the radiation interaction modeled here is only a crude approximation, but is satisfactory for practical purposes since the emission and absorption of the gas tend to cancel each other out [25]. Also, scattering is unimportant for very small particles. After performing the indicated partial differentiation, the rearrangement of the resulting expression leads to a set of four non-linear integral equations of the Volterra type,

$$\sum_{k=1}^{\infty}\frac{\partial}{\partial\tau}\int_0^{\tau}(Z_{00}-1)\,\mathrm{e}^{-\beta_k^2(\tau-\lambda)}\,\mathrm{d}\lambda=-\frac{\nu\Phi_\kappa}{2\mathscr{D}}\left[\frac{U_{00}Z_{00}}{X_{00}}+\Omega_1U_{00}\right]+\tfrac{1}{2}\varphi_{\mathrm{eff}}\Omega_2\left[1-\left(\frac{T_0-T_\infty}{T_\infty}Z_{00}+1\right)^4\right] \tag{85}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\partial}{\partial\tau}\int_0^{\tau}Z_{01}\,\mathrm{e}^{-\beta_k^2(\tau-\lambda)}\,\mathrm{d}\lambda=\frac{\Phi_\kappa}{2}\left[\left(\frac{\tilde{U}_{00}-1}{1-\mathrm{e}^{-\tilde{U}_{00}}}\right)\tilde{Z}_{00}U_{01}+C_0\tilde{U}_{00}\right]\mathrm{e}^{-\tilde{U}_{00}}-\frac{\nu\Phi_\kappa}{2\mathscr{D}}\Omega_1U_{01}\\-2\varphi_{\mathrm{eff}}\Omega_2\left(\frac{T_0-T_\infty}{T_\infty}\right)\left[\frac{T_0-T_\infty}{T_\infty}Z_{00}+1\right]^3Z_{01}\end{aligned} \tag{86}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\omega_k\hat{C}_kJ'_{3/2}(\omega_k)\frac{\partial}{\partial\tau}\int_0^{\tau}Z_{11}\,\mathrm{e}^{-\omega_k^2(\tau-\lambda)}\,\mathrm{d}\lambda=4\Phi_\nu\,\hat{Pr}\,\hat{S}'(\tau,1)+\hat{G}'_{\mathrm{ss}}(\tau,1)+Z_{11}(\tau)\\-\Phi_\kappa\left[\frac{2Z_{11}}{2-\tilde{U}_{00}}-\frac{C_1\tilde{U}_{00}^3\,\mathrm{e}^{-\tilde{U}_{00}}}{2(2-\tilde{U}_{00})}-\frac{\nu\Omega_1}{\mathscr{D}}U_{11}\right]+4\varphi_{\mathrm{eff}}\Omega_2\left(\frac{T_0-T_\infty}{T_\infty}\right)\left[\frac{T_0-T_\infty}{T_\infty}Z_{00}+1\right]^3Z_{11}\end{aligned} \tag{87}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\gamma_k\hat{E}_kJ'_{5/2}(\gamma_k)\frac{\partial}{\partial\tau}\int_0^{\tau}Z_{21}\,\mathrm{e}^{-\gamma_k^2(\tau-\lambda)}\,\mathrm{d}\lambda=4\Phi_\nu\,\hat{Pr}\,\hat{D}'(\tau,1)+\hat{Q}'_{\mathrm{ss}}(\tau,1)+2Z_{21}(\tau)\\-\Phi_\kappa\left[\frac{(24-6\tilde{U}_{00})Z_{21}+C_2\,\mathrm{e}^{-\tilde{U}_{00}}}{\tilde{U}_{00}^2-6\tilde{U}_{00}+12}-\frac{\nu\Omega_1}{\mathscr{D}}U_{21}\right]+4\varphi_{\mathrm{eff}}\Omega_2\left(\frac{T_0-T_\infty}{T_\infty}\right)\left[\frac{T_0-T_\infty}{T_\infty}Z_{00}+1\right]^3Z_{21}\end{aligned} \tag{88}$$

where φ_{eff} $(=\varphi/R_\varphi)$ the effective absorptivity. The superscript 'prime' indicates the derivative with respect to the radial coordinate. The dimensionless constants Ω_1 and Ω_2 are the inverse Jacob number and the inverse conduction–radiation parameter respectively, i.e.,

$$\Omega_1=\frac{\mathscr{H}_{\mathrm{fg}}}{C_{\mathrm{p}}(T_0-T_\infty)},\quad \Omega_2=\frac{\sigma T_\infty^4}{\hat{\kappa}\varphi(T_0-T_\infty)} \tag{89a,b}$$

in which C_{p} is the heat capacity at constant pressure.

All of the above integral equations have a special feature in that the kernels exhibit convolutional structure. Furthermore, they are uncoupled in a sense that they can independently be solved sequentially using the numerical procedure in Nguyen [23]. Due to the nonlinearities, iteration is required in advancing the solution from one time-step to the next.

The vaporization rate, $\dot{m}$, may be found by integrating the radial interfacial velocity over the drop surface so that

$$\begin{aligned}\dot{m}&=-\frac{\pi\hat{\varrho}\hat{\alpha}}{D}\frac{\mathrm{d}D^2}{\mathrm{d}\tau}=\pi D\varrho\nu\int_{-1}^{1}[U_{00}+\varepsilon(U_{01}+U_{11}P_1(\bar{\mu})+U_{21}P_2(\bar{\mu}))]\,\mathrm{d}\bar{\mu}\\&=2\pi D\varrho\nu(U_{00}+\varepsilon U_{01})\end{aligned} \tag{90}$$

in which only the uniform terms (U_{00} and U_{01}) survive whereas the net contribution from P_1 and P_2 terms are identically zero. Alternatively, equation (90) can be rewritten as

$$\frac{\mathrm{d}(\ln D^2)}{\mathrm{d}\tau}=-\frac{2\Phi_\kappa}{\Phi_{c_{\mathrm{p}}}}(\tilde{U}_{00}+\varepsilon\tilde{U}_{01}). \tag{91}$$

Due to the time-dependence of the perturbation variable, ε, we shall need another equation to complement equation (91). One way of doing that is to write the Newton's second law for the motion of the drop,

$$\frac{\mathrm{d}U_\infty}{\mathrm{d}\tau}+\Phi_\mu\,\hat{Pr}\,F_{\mathrm{D}}U_\infty=\frac{g\Phi_\mu\,\hat{Pr}}{8\hat{\nu}^2}\left(\frac{1}{\Phi_\rho}-1\right)D^2. \tag{92}$$

The expressions derived thus far are primarily for an evaporating droplet; however, the treatment is general and is well applicable to a condensing droplet. In the latter case, one needs to have a compatibility condition for the condensation process to replace the interfacial thermodynamics relating temperature and concentration.

4. SAMPLE CALCULATIONS AND DISCUSSION

To the authors' knowledge, very few studies conducted in the past have addressed the transient nature of the interfacial quantities associated with an evaporating droplet. More often than not, they are only concerned with the physical parameters that are of practical interest and overlooked the detailed mechanisms particularly at the interface. In order to demonstrate the usefulness of our present formulation, we have calculated the time evolution of the interfacial temperature, concentration, and velocity of an Octane–Air system. These fundamental quantities undoubtedly play a major role in the phase change process. The selection of octane as a test liquid is just a matter of convenience because of the availability of its thermophysical properties in the literature. Table 1 below gives a listing of the physical properties of various fuels.

The physical system we now analyze consists of a droplet at an initial temperature of 298 K moving at a velocity equivalent to $Re = 0.1$ in an unbounded gaseous medium held at 380 K and both media are under the influence of an electric field whose strength is chosen such that it produces a maximum velocity equal to that generated by translation ($\mathscr{W} = 1.0$). With the system so defined, the flow patterns inside and outside the drop are given in Fig. 1 at two different instances ($\tau = 0.1$ and 0.5). As clearly shown in the figure, the electric field modifies the internal flow by doubling the number of closed contours which are nearly symmetrical about the y-axis, and almost equal in sizes. These loops may run from the pole toward the equator or vice versa depending on the electrical properties of the two phases. To be specific, the former is appropriate for a positive $\mathscr{W}$ whereas the latter is suitable for a negative $\mathscr{W}$. Outside the drop, there are two distinct regions, both of which are essentially unaffected by the electric field. Within the inner region, all the streamlines emanating from the drop surface would redirect themselves into the direction of the freestream. On the other hand, the motion in the outer region resembles the flow past a bluff object with the dividing streamline considered to be the shape of the body. A detailed examination of the figure indicates that the dividing streamline tends to be pushed away from the drop surface as the elapsed time increases until it reaches an equilibrium position where the distance from the drop center is controlled by the intensity of the evaporation process. Due to the non-vanishing initial interfacial velocity, the starting location of the dividing point would not be on the surface as opposed to the case of flow without mass transfer. For $\mathscr{W} = -25$, recirculation is established in the wake as shown in Fig. 1(c). A more detailed discussion on the various flow structures including the recirculating bubble in the wake and the effects of the electric field on the dividing streamline are given in Nguyen and Chung [19].

In Fig. 2(a) we plotted the time evolution of the interfacial temperature of an octane drop under evaporation in air. The data reveal that the temperature is almost uniform over the drop surface at any time since the coefficients of the angular terms (Z_{11} and Z_{21}) are much smaller than the leading order solution (Z_{00}), that attains a time-invariant state with a value 25% lower than the initial temperature of the droplet. This aspect of the interfacial temperature would suggest a possible simplification to the problem which is to assume an angular-dependent-free condition for the interfacial temperature. Of course, this is true as long as the translational velocity is limited to the creeping regime. A separate study is currently underway to assess this assumption at high Reynolds numbers, and the findings will be reported in a later note. It should be mentioned that for the system under consideration, the initial interfacial temperature is close to the initial drop temperature, but this is not always the case in practice. In fact, the actual value is dictated by the thermal conductivities of the fuel as well as the ambience. The graph also illustrates, as one now recognizes, that the two-phase transport problem should be treated as a conjugate problem rather than an internal/external problem because the drop fails to keep its interior at the interfacial temperature.

Since a reduction in dimensionless temperature corresponds to an increase in dimensional temperature, the concentration of the drop material, as governed by the equation describing the phase change phenomena at

Table 1. Physical properties of air–fuels

	Air	n-Octane	n-Decane
ϱ, g cm^{-3}	4.12×10^{-3}	0.661	0.708
κ, cal s^{-1} cm^{-1} K^{-1}	5.12×10^{-5}	2.65×10^{-4}	2.71×10^{-4}
μ, g s^{-1} cm^{-1}	5.52×10^{-5}	3.83×10^{-3}	4.53×10^{-3}
C_p, cal g^{-1} K^{-1}	0.241	0.503	0.496
$\mathscr{D}$, cm^2 s^{-1}	—	0.142	0.105
$\mathscr{H}_{fg}$, cal g^{-1}	—	76.65	76.92
T_b, °C	—	125.66	174.12
W, g mol^{-1}	29.0	114.224	142.276
χ_1†	—	15.9426	16.0114
χ_2†	—	3120.29	3456.80
χ_3†	—	−63.63	−78.67

† Antoine constants used in equation (38).

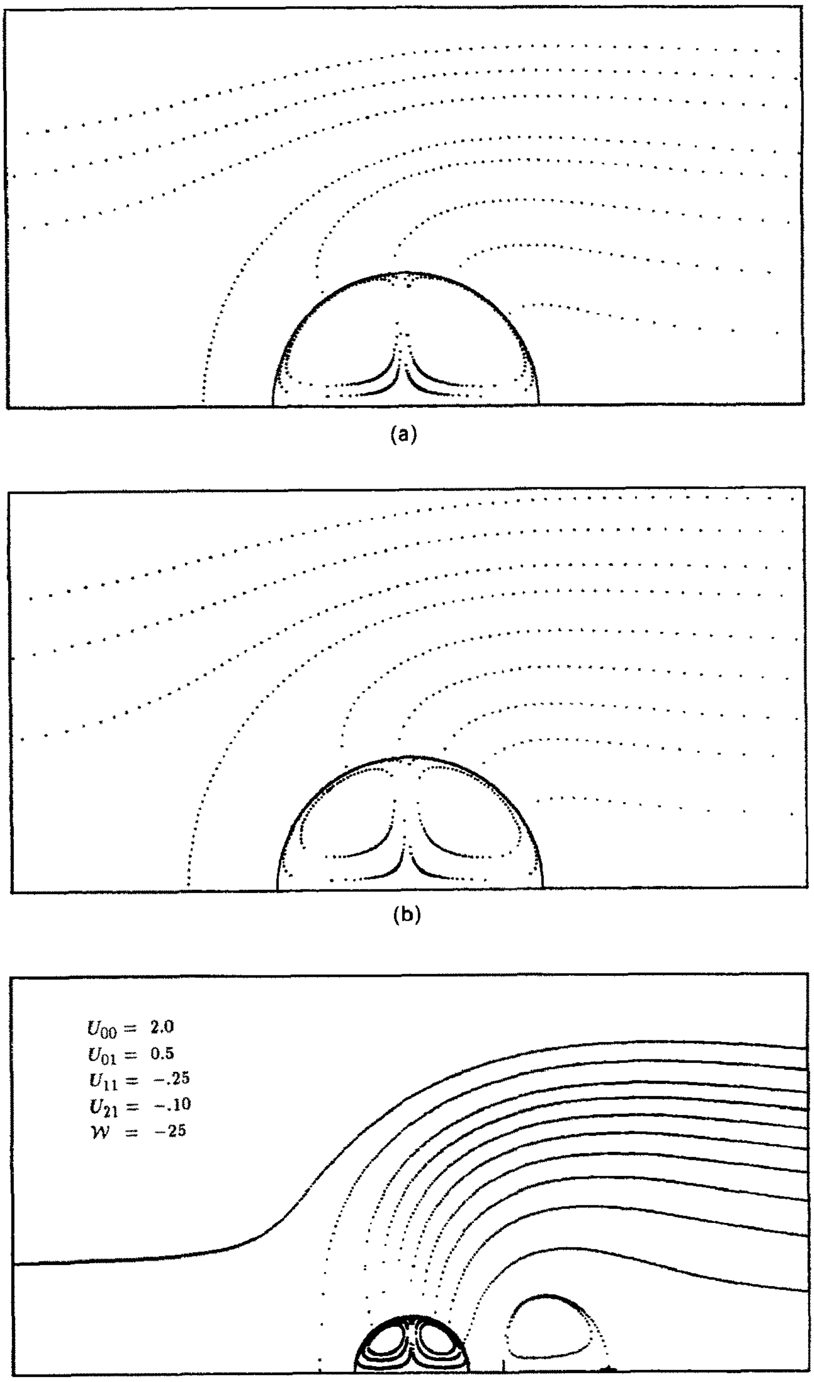

FIG. 1. (a) Stream function contour of an octane drop vaporizing in air at $\tau = 0.1$. (b) Stream function contour of an octane drop vaporizing in air at $\tau = 0.5$. (c) Stream function contour inside and outside a drop.

the gas–liquid contacting surface, consequently increases as time proceeds. Realistically, the phase change occurs so fast that the combined diffusion and convection could not keep-up with the gasification rate, hence a material build-up of the evaporating species follows as seen in Fig. 2(b). Although the angular variation of the concentration of the vaporizing species in the gas phase is negligibly small, the interfacial velocity is strongly angular-dependent as demonstrated in Fig. 2(c). Except the term responsible for the contribution due to the electric field, whose role is important primarily in the early stage of the evaporation process, U_{01} and U_{11} each contributes more than 10% of the leading order velocity especially at small time.

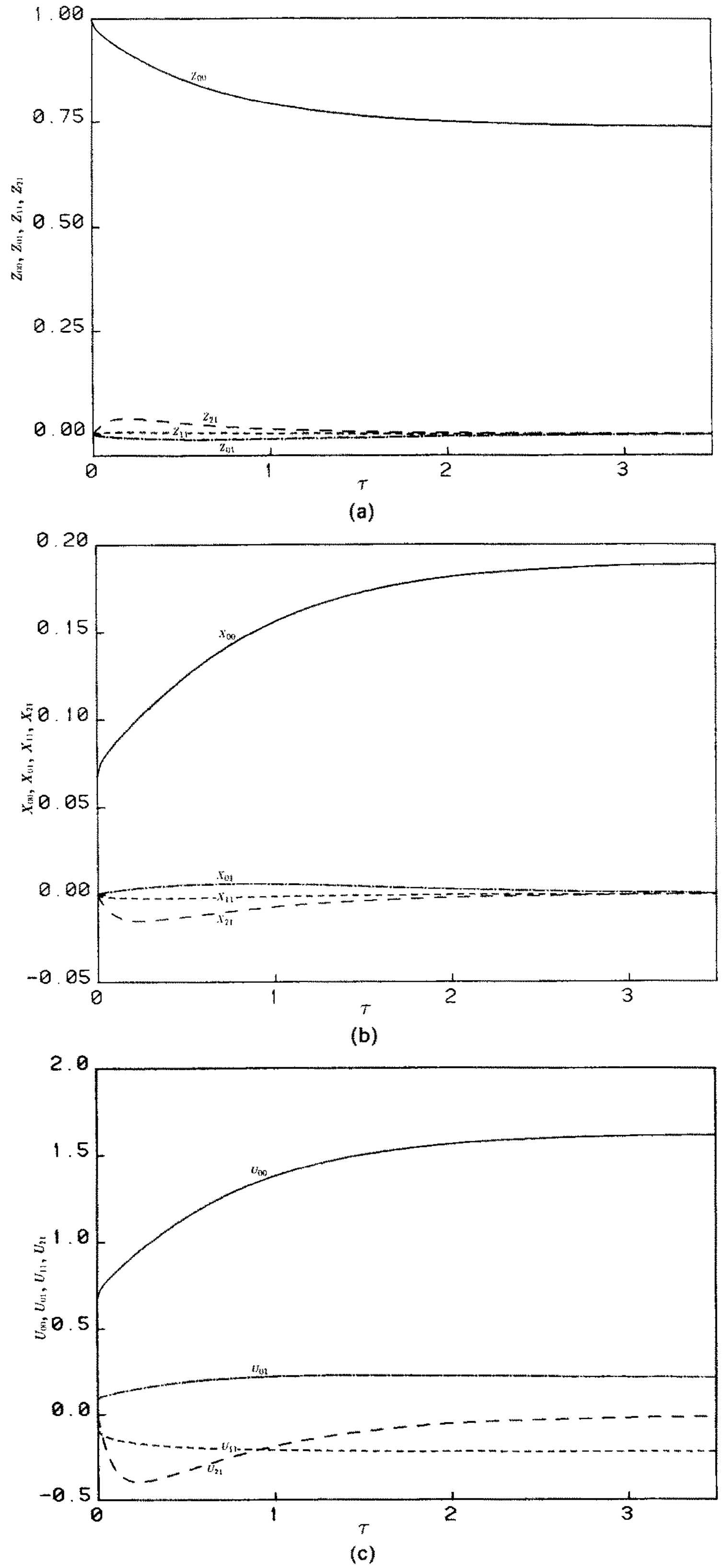

FIG. 2. (a) Surface temperature of a vaporizing octane drop. (b) Surface concentration of a vaporizing octane drop. (c) Surface velocity of a vaporizing octane drop.

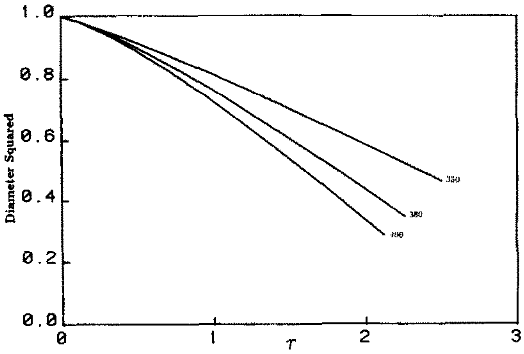

FIG. 3. Instantaneous diameter of an octane drop.

In Fig. 3, the square of the dimensionless drop diameter is plotted against time at three ambient temperatures: 350, 380, and 400 K. In all the cases, the d^2-Law is recovered as expected instead of the $d^{3/2}$-Law for a drop exposed to a strong convective environment because the rectilinear motion of the drop is one order-of-magnitude smaller than its radial counterpart. Accompanying Fig. 3 is a plot, Fig. 4, of the drop size history for different fuels having considerable difference in boiling temperatures. The octane drop evaporates faster mainly because it has much lower boiling point and higher diffusion coefficient.

The results presented thus far did not take into account the exchange of radiant energy because the temperature is considered to be relatively low for thermal radiation to have an appreciable impact. However, when the drop is subjected to a combustion environment where the temperature is of the order of thousand degrees, the effects of radiation may become significant. As a test case for our current model, we present the result regarding the influence of thermal radiation upon the evaporation rate as depicted in Fig. 5 for a drop

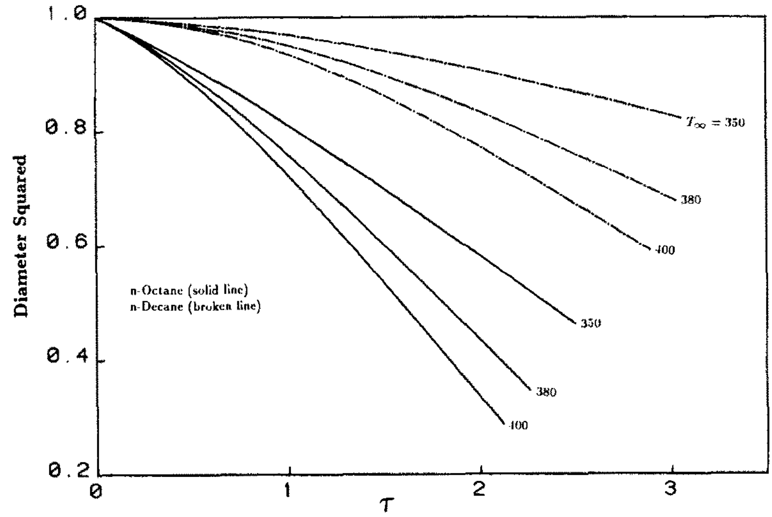

FIG. 4. Drop size history of various fuels.

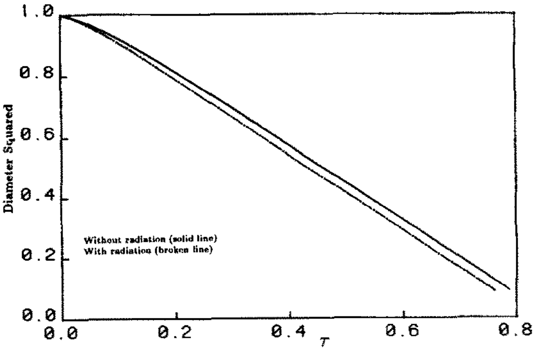

FIG. 5. Effect of radiation on evaporation rate.

initially held at room temperature (298 K) exposed to an ambience at 2000 K. Here we assumed, as typical values, the absorptivity of 0.1 and 0.001 for a small drop and surrounding gas, respectively. From the figure, the effect of radiation is simply to shorten the heating-up period, hence the drop lifetime, without changing the slope of the d^2-Law. Even for an ambience as high as 2000 K, the drop would disappear only about 5% of the total residence time quicker.

Although the contribution of radiation in the test case is small, one should be aware of its role as the temperature increases due to the fact that radiation is proportional to T^4. Nonetheless, when radiation is important the current analysis would offer a great advantage in that it can handle the nonlinear boundary conditions without any additional computational difficulties. Furthermore, the computing time is virtually unchanged so that one would be allowed to concentrate on other effects associated with the phenomena at the interface such as the surfactant effects, for example. In any case, a complete physicochemical run consumed not more than 150 s of CPU time on an IBM-3090 for 3500 time steps with an increment of 0.001. Unlike the work of Gogos *et al.* [26] where the equations seem to couple together, and hence the solutions require a simultaneous simulation that would be costly in terms of the computing time, the present model eliminates them. This was accomplished by solving the integral equations for the interfacial parameters before the global equations are addressed. Such eliminations were made possible since they were only introduced through the boundary separating the two phases.

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